Invited Lecture

Mathematics: A Code for Interdisciplinary Dialogues

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ABSTRACT In this talk, I will introduce mathematics as a code for interdisciplinary dialogues through a story on infinity.

Keywords: Mathematics; Infinity; Interdisciplinary dialogue.

1. Overture

What is the answer to the following question?

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=?$$

How about this solution:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1+\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right).$$

$$\text{Let } S=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

$$\text{Then } S=1+\frac{S}{2}, \text{ so } S=2.$$

Seems to be good.

What is the answer to the following question?

$$1+2+4+8+\cdots=?$$

How about this solution:

$$1+2+4+8+\cdots=1+2(1+2+4+\cdots).$$

$$\text{Let } S=1+2+4+8+\cdots$$

$$\text{Then } S=1+2S, \text{ So } S=-1.$$

I have applied similar strategy as before. However, it seems to be no good.

What's the difference between these two infinite situations? What is happening in infinity?

I have a circle and inscribed regular polygons infinitely many (Fig. 1).

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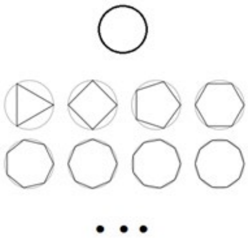


Fig. 1 A series of inscribed regular polygons

The regular polygons approach the circle. And the lengths of regular polygons approach the length of the circle.

Now I have another series of curves (Fig. 2).

$$f_n: [0,2\pi] \to \mathbb{R},$$

$$f_n(x) = \frac{1}{n}\sin nx$$

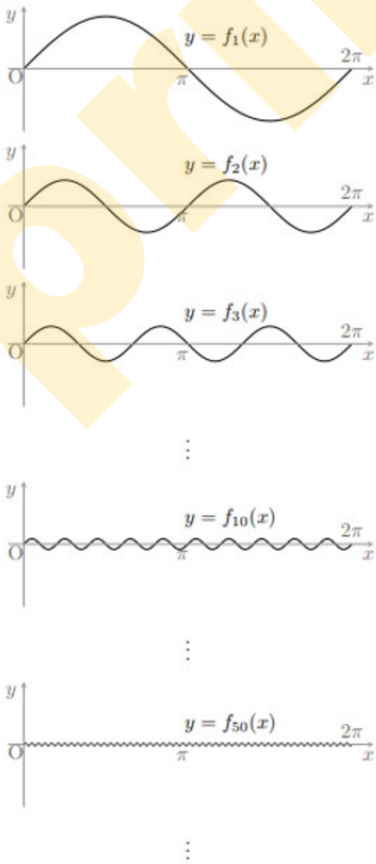


Fig.2 A series of curves

The curves approach the line segment over $[0, 2\pi]$. However, the lengths of the curves do not approach the length of the line segment.

What is the difference between these two infinite situations? What is happening in infinity? We are curious about infinity.

Mathematics seems to be a reasonable language and grammar for infinity. So it is interesting to try an interdisciplinary dialogue on infinity based on mathematics.

At first, we note that a difference between rational and irrational numbers is finiteness and infiniteness. A real number r is rational if and only if r can be presented as a finite simple continued fraction. In other words, a real number r is irrational if and only if r can be presented as an infinite simple continued fraction. For examples, I have some simple continued fractions.

$$1$$

$$1+\frac{1}{1}$$

$$1+\frac{1}{1+\frac{1}{1}}$$

$$1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}$$

These are finite, so rational numbers.

$$1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}$$

This is infinite, so is an irrational number. Infinity is in irrational numbers.

Hyperbolic geometry (Fig. 3) and elliptic geometry are examples of non-Euclidean

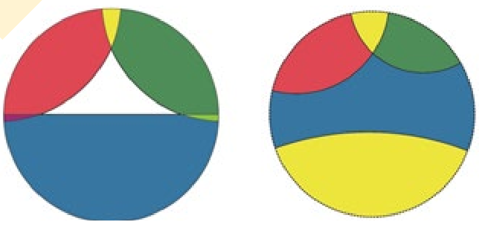


Fig. 3 Properties of hyperbolic geometry

geometry. It is the parallel axiom which distinguishes Euclidean geometry from the non-Euclidean. Parallel axiom without infinity is not so meaningful in mathematics. Infinity is in non-Euclidean geometry.

George Pólya has proved that there are 17 wallpaper patterns. Using Pólya's argument, we can easily show that there are 7 frieze patterns. We note that mathematical classification of wallpaper and frieze patterns needs symmetries and infinity. Let us consider some examples.

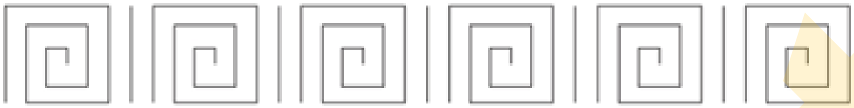


Fig. 4 A frieze pattern with translation symmetry

This frieze has translation symmetry (Fig. 4). It means that we can move the frieze to the right or to the left without any change of the pattern. To say so, we have to assume that the frieze be repeated infinitely in the right and in the left.

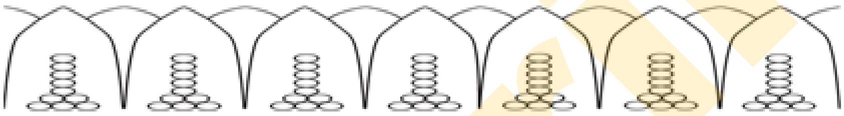


Fig. 5 A frieze pattern with reflection symmetry

This frieze has reflection symmetry (Fig. 5). To say that the following vertical lines are axes of symmetry (Fig. 6), this frieze is assumed to be repeated infinitely in both directions.

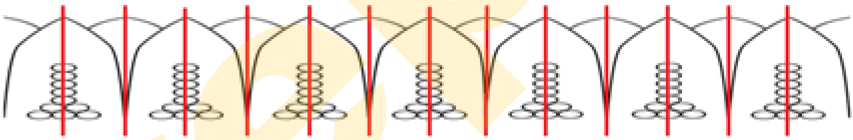


Fig. 6 Axis for reflection symmetry

This frieze has 180°-rotation symmetry (Fig. 7).

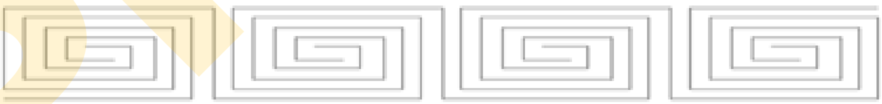


Fig. 7 A frieze pattern with rotation symmetry

To say that the following points are axes of symmetry (Fig. 8), we have to assume that the frieze be repeated infinitely in both directions.

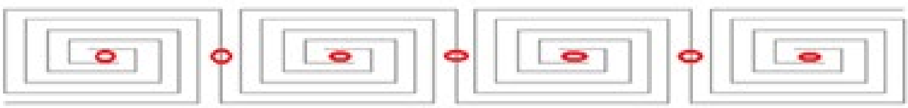


Fig. 8 Axis for rotation symmetry

The same is true in the case of wallpaper. Wallpaper is assumed to be repeated infinitely in four directions. Mathematical approach to patterns requires infinity.

2. Infinity in Music

In *School of Athens* by Raffaello, there are some mathematicians. Pythagoras is one of them. Raffaello presents him with the tetractys. In tetractys, there are some rational numbers: three quarters, two thirds, one half, and eight ninths. These rational numbers are the basis of Pythagorean Scale in music. Those numbers are on the neck of guitar.

About two thousand years after Pythagoras, the tetractys was replaced by the Euler Tonnetz. Euler Tonnetz represents the equal temperament which is based on irrational numbers. Equal temperament uses irrational numbers in substitution for rational numbers. A basic difference between Pythagorean scale and equal temperament is finiteness and infiniteness. Infinity is in music theory.

These are traditional instruments of China, Korea, Vietnam, and Russia as shown in Fig. 9.

Fig. 9 Old traditional music instruments

Today's versions of these instruments are shown in Fig. 10. Compare the positions of frets. They are not same.

Fig.10 New traditional instruments

How about here? The positions of frets are exactly same as in guitar. In guitar, the positions of frets form a curve (Fig. 11).

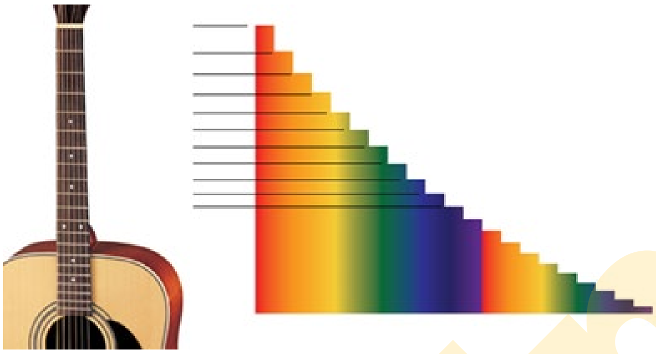
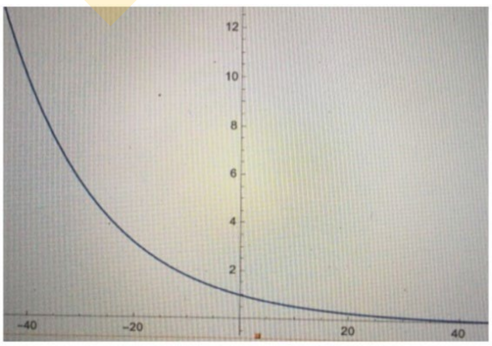


Fig. 11 A curve in guitar

The same curve can be found in piano and in pan flute (Fig. 12).

Fig. 12 Curves in piano and pan flute

The curve is given by this function (Fig. 13). Irrational number is in musical instruments. Infinity is there.



$$y = \left(\frac{1}{\sqrt[12]{2}}\right)^{x}$$

Fig. 13 Mathematical curve

Geomungo is one of the traditional instruments of Korea. Geomungo can be seen in paintings in tombs of 5th century, and in paintings of 18th century of Korea. Geomungo has a long history and is popular in Korea. This is an old geomungo (Fig. 14). Let us pay attention to positions of frets. There seems to be no mathematics there.

Fig. 14 Old geomungo

However, in today's version seen (Fig. 15), the positions of frets are similar to those in guitar. The irrational numbers are in geomungo. The infinity is even in geomungo.

Fig. 15 Frets in new geomungo and guitar

3. Infinity in Paintings

To understand and to classify the 7 frieze patterns, 17 wallpaper patterns, or Escher's patterns, we need symmetry and infinity.

Max Bill was educated at Bauhaus, and has served as a director of a design school. He has had a dream of new form of art based on mathematics: "I am convinced it is possible to evolve a new form of art in which the artist's work could be founded to quite a substantial degree on a mathematical line of approach to its content." A substantial degree on a mathematical line of approach must require infinity. There is infinity in Max Bill's paintings. Maldonado has also served the same design school as a director. There is infinity in Maldonado's works. Infinity is in art.

Le Corbusier thought that a house is a machine for living in. He wanted the various postures of human being to be considered in architecture. His "Modulor" gave him a solution. Le Corbusier believed that there are many divine proportions in human body.

The divine proportion is an irrational number. Infinity is in “Modulor.” Infinitely is in architecture.

4. Infinity in Literatures

Dostoevsky mentions non-Euclidean geometry. Infinity in his novel. Tolstoy mentions infinitesimal, Newton’s law of gravity, continuity, and discontinuity in *War and Peace*. Infinity is in Tolstoy’s novel.

The Man without Qualities is a novel of Musil. Musil thought that mathematics is the mother of natural science. Mathematics is important in this novel. In fact, the lead character of the novel is Ulrich. He is a mathematician. And one of the themes of the novel is mathematics and mysticism. Infinity should be there.

In Broch’s novel *The Sleepwalkers*, the crisis of foundations of mathematics in the early 20th century is the basic background. The infinity is the key and the essence in the foundation of mathematics. *The Aleph* is a novel of Borges. In mathematics, $\aleph$ denotes the trans-finite cardinality. In the novel, the aleph is a point in space that contains all other points in the world. This is quite similar to the definition of an “infinite set” in mathematics. In mathematics. an infinite set contains infinitely many proper subsets which are equivalent to itself.

Queneau was a member of Oulipo. To him, mathematics was a source of inspiration.

Queneau has proposed *The Foundations of Literature* in the same spirit of *The Foundations of Geometry* by David Hilbert. According to Hilbert, point, line, plane can be replaced by table, chair, and drinking glass. In *The Foundations of Literature*, point, line, and plane are replaced by word, sentence, and paragraph respectively. The parallel axiom in *The Foundations of Literature* is this: “A sentence having been given, and a word not belonging to this sentence, in the paragraph determined by the sentence and this word, there exists at the most one sentence including this word which has no other word in common with the first given sentence.” Queneau eventually claimed that every sentence includes an infinity of words: one perceives only a very few of them, the others being in the infinite or being imaginary. In a book by Queneau, there are 10^{14} sonnets. It is not possible for anyone to read all of those poems. 10^{14}sonnets are finitely many in mathematics, but infinitely many in literature.

In Pynchon’s novel *Against the Day*, some deep mathematics are mentioned quite seriously. Infinity is there. Szymborska was interested in π, the circumference rate. She has written a poem under the title π. π became a poem. The infinity of π became a poem. Infinity is in literature.

5. Infinity in Philosophy

In Plato’s *Meno*, Socrates, Meno, and a servant of Meno have a dialogue. The theme of the dialogue is the length of a side of a square with area 8, which is an irrational number. At first, the servant thought that he knew the length. During dialogue, he became to know that he didn't know the length. At last, the servant became to know himself. Infinity is in Plato's dialogues. Aristotle, in his book *Physics*, discusses Zeno's paradoxes. Without infinity, Zeno's paradoxes do not make any sense.

Newton and Leibniz developed a theory of infinity, called "differential calculus." Their philosophical approaches, however, were quite different. Leibniz thought that infinities and infinitesimals are fictions after all, though well-founded ones. To Leibniz, differential calculus was a logical fiction. Philosopher Berkeley did not accept the theory of Newton and Leibniz. In one of his books, Berkeley refuted differential calculus quite critically. Differential calculus is not only mathematics, but also philosophy. Differential calculus is a theory on infinity.

David Hume, a philosopher of empiricism, did not accept the infinite divisibility. His attitude for infinity was totally different from the conventional mathematics. Infinity is in philosophy.

Karl Marx has written about 850 pages of manuscripts on differential calculus. Differential calculus is a basic theory of motion. It is possible for him to try to have a basic principle of social change from differential calculus. It is also probable that Marx has tried to get a theoretical foundation of communism from differential calculus.

Infinity is even in politics.

Theology discusses existence, love, perfection, greatness, immortality of God. Without infinity, such discussions are not possible. Infinity is in theology.

6. Finale

Infinity is in music, art, literature, philosophy, politics, and theology. How about infinity in mathematics? Euclid, in his book *The Elements*, proposed the parallel axiom. Infinity is there. Euclid also proved that there are infinitely many prime numbers. Infinity is in Euclidean mathematics.

Archimedes has obtained the bound for π from a regular polygon of 96 sides. It is quite clear that Archimedean curiosity on circumference rate was not stopped by his bound. He might have imagined the infinity of π much more. Infinity is in Archimedean mathematics.

As far as infinity is concerned, volume of sphere as well as π is interesting. Archimedes was so glad to have this ratio which exists among the volumes of cone, sphere, and cylinder (Fig.16).

Liu Hui, a Chinese mathematician, tried to compute the volume of sphere using the sphere inscribed in the intersection of two cylinders of equal radius at right angles (Fig.17).

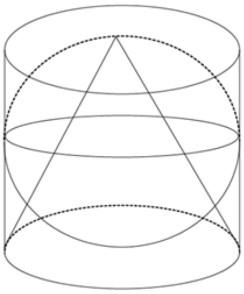


Fig. 16 Cone and sphere in cylinder

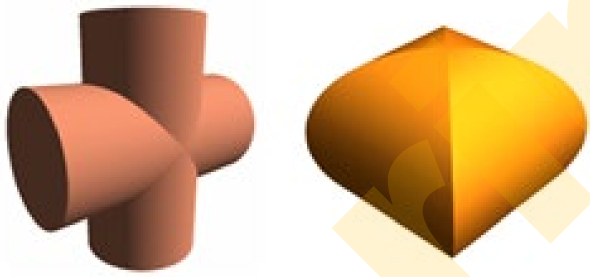


Fig. 17 Intersection of two cylinders

Unfortunately, he was not successful. Liu Hui, however, obtained the bound for π which was as sharp as Archimedean bound.

$$3.141 < \pi < 3.142.$$

It was Zu Chongzhi who challenged the problem again many years after Liu Hui. Zu Chongzhi was successful in getting $\frac{4}{3}\pi r^3$. He used so called 'Cavalieri's principle'. However, we know that Zu Chongzhi was more than one thousand years older than Cavalieri.

Zu Chongzhi, on the other hand, obtained the following bound for π.

$$3.1415926 < \pi < 3.1415927.$$

This bound was so good as not to be sharpened more for the next many hundred years. Probably, Zu Chongzhi's curiosity on π was not stopped by this nice bound. He might have imagined the infinity of π much more. There was infinity in Chinese mathematics of many years ago.

Bolzano has struggled with infinity. Bolzano eventually came up with a mathematics of infinity.

Cantor developed a mathematics of infinity. How many points are there in the one-dimensional figure (Fig. 18)?

$(0,1)$

Fig. 18 Points on a line

How many points are there in the two-dimensional figure?

$(0,1)\times(0,1)$

Fig. 19 Points on a square

Cantor proved that these two infinite sets are equivalent. They have the same cardinality. After proving this fact, Cantor shouted:

"I see it, but I don't believe it."

In infinity, Cantor's heart had difficulty in following his own head.

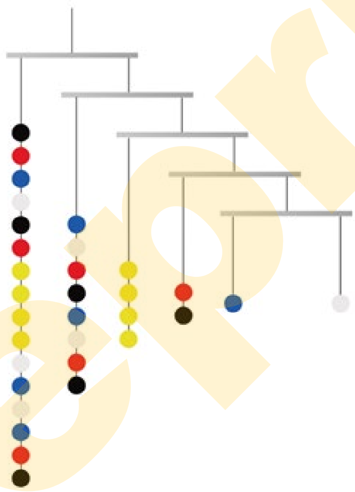


Fig. 20 Mobile for a geometric sequence

This mobile (Fig. 20) can be extended as much as we wish. This mobile says that $2^n < 2^{n+1}$.

What will happen if n goes to infinity? Cantor was surprised with the following equality:

$$2^{\aleph_0} = 2^{\aleph_0+1}.$$

Clearly $n < 2^n$. What will happen if n goes to infinity?

$$\aleph_0 < 2^{\aleph_0}.$$

Cantor was also surprised with the above inequality. Now it is easy to give natural numbers between n and 2^n. What will happen if n goes to infinity? What are there between these two trans-finite cardinalities $\aleph_0$ and $2^{\aleph_0}$?

Unfortunately, Cantor could not answer this his own question until his death. It was Gödel who challenged this problem again some years after Cantor's death. Gödel's solution eventually became the continuum hypothesis. Gödel, furthermore, proved the incompleteness theorems. The incompleteness theorems revealed unexpected properties of axiomatic mathematics involving infinity. For example, a consistent system of axioms involving infinity cannot be complete. In such a system, there is a fact that cannot be proved to be a fact by axiomatic mathematics. There are non-provable facts as well as provable facts. In other words, there are non-provable non-facts as well as provable non-facts.

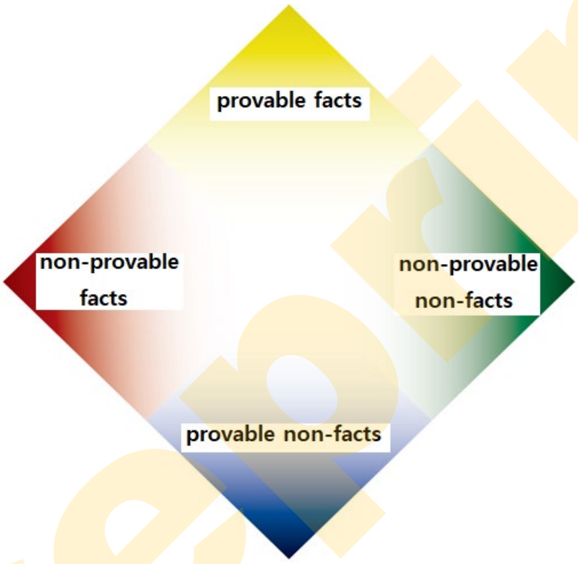


Fig. 21　Provable and non-provable

Furthermore, there is a big area which is beyond the axiomatic mathematics.

Zermelo was interested in axiom of choice which is an axiom of infinity. Based on axiom of choice, Banach and Tarski have proved a theorem called 'Banach-Tarski paradox.' Banach-Tarski paradox says that axiom of choice transforms an apple into two apples of same volume (Fig. 22).

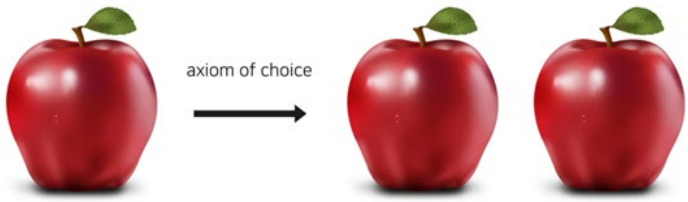


Fig. 22　Axiom of choice

Now let me close my talk. Mathematics says that infinity is mysterious. However, infinity is imaginable through mathematics. Mathematics might be the best language and grammar for infinity. Mathematics could be a code for interdisciplinary dialogues on infinity.

References

H. Shin (2020). *Symphonia Matematica* (in Korean). Chengju: Mathesign.